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MULTIVALUED PERTURBATIONS OF m-ACCRETIVE DIFFERENTIAL INCLUSIONS

BY

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ABSTRACT

Given an m-accretive operator A in a Banach space X and an upper semicontinuous multivalued map F: $[0,a] \times X \to 2^X$, we consider the initial value problem

$$u' \in -Au + F(t,u) \quad \text{on } [0,a], \quad u(0) = x_0.$$

We concentrate on the case when the semigroup generated by $-A$ is only equicontinuous and obtain existence of integral solutions if, in particular, X^* is uniformly convex and F satisfies

$$\beta(F(t,B)) \leq k(t)\beta(B) \quad \text{for all bounded } B \subset X$$

where $k \in L^1([0,a])$ and β denotes the Hausdorff-measure of noncompactness. Moreover, we show that the set of all solutions is a compact R_δ-set in this situation. In general, the extra condition on X^* is essential as we show by an example in which X is not uniformly smooth and the set of all solutions is not compact, but it can be omited if A is single-valued and continuous or $-A$ generates a C_0-semigroup of bounded linear operators.

In the simpler case when $-A$ generates a compact semigroup, we give a short proof of existence of solutions, again if X^* is uniformly (or strictly) convex. In this situation we also provide a counter-example in $\mathbb{R}^4$ in which no integral solution exists.

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1. Introduction

We consider the initial value problem

$$(1) \qquad u' \in -Au + F(t,u) \quad \text{on } J = [0,a], \quad u(0) = x_0$$

in a real Banach space X, where A is m-accretive and F is a multivalued map. Given $x_0 \in \overline{D(A)}$, we look for (integral) solutions of (1). Here u is called a solution of (1) if u is the integral solution (see Section 2 below for the definition) of the quasi-autonomous problem

$$(2) \qquad u' \in -Au + w(t) \quad \text{on } J, \quad u(0) = x_0$$

with some

$$w \in \mathrm{Sel}(u) := \{v \in L^1_X(J) \colon v(t) \in F(t, u(t)) \text{ a.e. on } J\}.$$

This problem has been studied by several authors. In [11], [12] and part III of [1] problem (1) is considered in a finite-dimensional Hilbert space and existence of solutions is obtained if (among other assumptions) F is continuous, or jointly measurable and lower semicontinuous in x, or measurable in t and upper semicontinuous in x, respectively. Under the last-mentioned conditions on F the same conclusion is proved in part IV of [1] for infinite-dimensional Hilbert spaces, given that A is of special type (as described at the beginning of Section 4 below); especially the semigroup generated by $-A$ is supposed to be compact. In all these cases integral solutions are in fact strong solutions, i.e. absolutely continuous and a.e. differentiable such that the inclusion in (1) holds a.e. on J. A short proof of the results mentioned so far can be found in Appendix A4 in [15]. In general, integral solutions may not be strong solutions; regularity results can be found e.g. in [4] and [9].

In [21] the authors obtain local integral solutions if X is a separable Banach space, the semigroup generated by $-A$ is compact and F is, in particular, lower semicontinuous. The proof given there is based on the known fact that the restriction of $\mathrm{Sel}\colon C_X(J) \to 2^{L^1_X(J)} \setminus \emptyset$ to any compact set has a continuous selection under these assumptions. Using "directionally continuous" selections of F it is shown in [8] that (1) admits a global integral solution u which satisfies the additional constraints $u(t) \in K$ on J for a given closed $K \subset \overline{D(A)}$, if X is a Banach space, $x_0 \in K$, $-A$ generates a compact semigroup and F is bounded lower semicontinuous with closed values and satisfies a natural subtangential condition. There the same approach is also used to prove connectedness of the solution set under the additional assumption that X^* is uniformly convex.

Viability results under time-dependent constraints in the upper semicontinuous case have been obtained in [7]. Based on [6], where single-valued continuous perturbations are considered, it is shown in [7] that (1) admits a mild solution satisfying additional constraints $u(t) \in K(t)$ on J if, in particular, $-A$ generates a compact semigroup and F satisfies a necessary subtangential condition.

Let us also note in passing that strong solutions of problem (1) with F satisfying a condition of dissipative type are obtained in [5].

In this paper we consider perturbations F being (weakly) upper semicontinuous in x, and we concentrate on the case when the semigroup generated by $-A$ is only equicontinuous. Under such conditions problem (1) has been studied in Chapter 3 in [28]; see Remarks 1 and 3 below for the precise assumptions used there.

Evidently u is a solution of (1) iff $u \in C_X(J)$ is a fixed point of $G := S \circ \mathrm{Sel}$, where (for fixed $x_0 \in \overline{D(A)}$) Sw denotes the unique integral solution of (2) corresponding to $w \in L^1_X(J)$. Hence we look for fixed points of G, where we always assume that F is defined on $J \times D$ with certain $D \supset \overline{D(A)}$ and, to avoid problems concerning the continuation of local solutions, we impose the growth condition

$$(3)\quad \|F(t,x)\| := \sup\{|y| \colon y \in F(t,x)\} \le c(t)(1+|x|) \quad \text{on } J\times D \text{ with } c \in L^1(J).$$

Once the global results are proved corresponding local versions follow easily; see Remarks 1 and 3 below. We start with the case when $-A$ generates a compact semigroup and obtain solutions of (1) if X^* is uniformly convex, F is weakly upper semicontinuous in x with closed convex values and the maps $F(\cdot,x)$ admit strongly measurable selections. If F is upper semicontinuous w.r. to x with compact values then strict convexity of X^* is sufficient. The extra condition on X^*, which can be dropped in the single-valued case $F = \{f\}$, is not a pure proof technical one in case of multivalued perturbations. This is shown by Example 1 below, where we have $X = \mathbb{R}^4$, F jointly usc but (1) has no integral solution for certain initial values.

In Section 4 we replace compactness of the resolvents of A by a compactness condition on F, namely

$$(4)\quad \beta(F(t,B)) \le k(t)\beta(B) \quad \text{a.e. on } J \text{ for all bounded } B \subset D \text{ with } k \in L^1(J).$$

By means of the estimate given in Lemma 4 below we then obtain existence of solutions of (1), again if X^* is uniformly convex (Theorem 2). This result is a considerable improvement of Theorem 3.6.1 in [28]. Let us also note that Theorem 2, if specialized to the single-valued case $F = \{f\}$, is an extension of Theorem 3.3 in [19] where f is assumed to be a compact map.

By the method of proof we also get compactness of the solution set. Since the latter is not true in general, as shown by Example 2 in which $F(t,x) \equiv C$ with some compact $C \subset X$, this method of proof is restricted to Banach spaces X having some additional property, unless A satisfies certain extra conditions. Such special situations in which the approach works in general Banach spaces are studied in Section 5. We start with the linear case and get a solution if $-A$ generates a C_0-semigroup of bounded linear operators and F satisfies the assumptions of Theorem 2. Next, we obtain the existence of strong solutions if $-A$ is replaced by g: $J \times X \to X$ being strongly measurable in t, continuous in x satisfying a growth condition like (3) as well as a condition of dissipative type. This extends one of the main results in [24]; see Remark 6 below.

In the final section we prove that the solution set is in fact a compact R_δ, i.e. the intersection of a decreasing sequence of compact absolute retracts; here we concentrate on the situation as described in Theorem 2 which is the more difficult one. In [20] it was shown that "absolute retracts" can be replaced by "contractible sets" in the definition of compact R_δ, and to obtain the result mentioned above we show that also "compact" can be weakened.

2. Preliminaries

In the sequel, X will always be a real Banach space with norm $|\cdot|$. Then $2^X \setminus \emptyset$ denotes the nonempty subsets of X, $\overline{B}_r(x)$ is the closed ball in X with center x and radius r, $B_r(x)$ denotes its interior and $\rho(x,B)$ is the distance from x to the set $B \subset X$. Given $J = [0,a] \subset \mathbb{R}$, we let $C_X(J)$ be the Banach space of all continuous u: $J \to X$ and $L^1_X(J)$ the Banach space of all strongly measurable, Bochner-integrable w: $J \to X$, both equipped with the usual norms which we denote by $|\cdot|_0$, respectively $|\cdot|_1$. Given an operator A: $X \to 2^X$, we let $D(A) = \{x \in X\colon Ax \neq \emptyset\}$, $R(A) = \bigcup_{x\in D(A)} Ax$ and $\mathrm{gr}(A) = \{(x,y)\colon x \in D(A), y \in Ax\}$ denote the domain, range and graph of A, respectively.

(i) Recall that A: $X \to 2^X$ is m-accretive if $R(I+\lambda A) = X$ for all $\lambda > 0$ and A is accretive, which means

$$(u-v, x-y)_+ \geq 0 \qquad \text{for all } x,y \in D(A),\ u \in Ax \text{ and } v \in Ay.$$

Here $(\cdot,\cdot)_+$ is given by $(z,x)_+ = \max\{x^*(z)\colon x^* \in \mathcal{F}(x)\}$ where $\mathcal{F}$: $X \to 2^{X^*} \setminus \emptyset$ denotes the duality map, i.e. $\mathcal{F}(x) = \{x^* \in X^*\colon x^*(x) = |x|^2 = |x^*|^2\}$; see e.g. §12.2 in [14].

If A is m-accretive, the resolvents $J_\lambda := (I+\lambda A)^{-1}$: $X \to D(A)$ are nonexpansive mappings, i.e. $|J_\lambda x - J_\lambda y| \leq |x-y|$ on $X \times X$, for all $\lambda > 0$. Conversely, if

$A: X \to 2^X$ is such that $R(I+\lambda A) = X$ and J_λ is single-valued and nonexpansive for every $\lambda > 0$, then A is m-accretive.

We will also use the following properties of J_λ, which hold if A is m-accretive. Given $x \in D(A)$ we have $|J_\lambda x - x| \le \lambda |y|$ for $\lambda > 0$, where y is any element of Ax; this implies $J_\lambda x \to x$ as $\lambda \to 0+$ on $\overline{D(A)}$. The resolvents satisfy the so-called resolvent identity

$$J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x\Big) \qquad \text{on } X \text{ for all } \lambda, \mu > 0.$$

Let us also recall some facts concerning the quasi-autonomous problem (2). Given any $w \in L^1_X(J)$ and $x_0 \in \overline{D(A)}$, initial value problem (2) has a unique integral solution u. This means $u: J \to \overline{D(A)}$ is continuous with $u(0) = x_0$ and

$$|u(t) - x|^2 - |u(s) - x|^2 \le 2\int_s^t (w(\tau) - y, u(\tau) - x)_+ d\tau \quad \text{for } 0 \le s \le t \le a$$

and all $(x, y) \in \mathrm{gr}(A)$. For fixed $x_0 \in \overline{D(A)}$, we let Sw denote the integral solution corresponding to $w \in L^1_X(J)$. Then $S: L^1_X(J) \to C_X(J)$ satisfies

$$|(Sw)(t) - (S\overline{w})(t)| \le |(Sw)(s) - (S\overline{w})(s)| + \int_s^t |w(\tau) - \overline{w}(\tau)| d\tau \text{ for } 0 \le s \le t \le a,$$

in particular S is Lipschitz of constant 1. In subsequent proofs it will happen that we know $w_n \rightharpoonup w$ in $L^1_X(J)$ and $Sw_n \to v$ in $C_X(J)$. We then wish to conclude $Sw = v$, but unfortunately this may fail even in finite dimensions (see Example 1 below). Nevertheless, it holds in the following situations.

Proposition 1: *Let X be a real Banach space, $A: X \to 2^X$ m-accretive and S defined as above. Let $(w_n) \subset L^1_X(J)$ be such that $w_n \rightharpoonup w$ in $L^1_X(J)$ and $Sw_n \to v$ in $C_X(J)$. Then $Sw = v$ holds if one of the following assumptions is satisfied:*

(a) *X^* is uniformly convex.*

(b) *X^* is strictly convex, $\beta(\{w_n(t): n \ge 1\}) = 0$ a.e. on J and $|w_n(t)| \le \varphi(t)$ a.e. on J for all $n \ge 1$ with some $\varphi \in L^1(J)$.*

We omit the proof, which is not difficult and relies on additional properties of the duality map depending on the extra conditions on X^*. The details can be found in [28], respectively [25].

If A is m-accretive, it generates a semigroup $\{T(t)\}_{t\ge 0}$ of nonexpansive mappings $T(t): \overline{D(A)} \to \overline{D(A)}$, given by the so-called exponential formula, i.e.

$$T(t)x = \lim_{n\to\infty} J^n_{t/n}x \quad \text{for } t \ge 0 \text{ and } x \in \overline{D(A)}.$$

Then $T(\cdot)x$ is the integral solution of (2) with $w \equiv 0$ and $u(0) = x$. $\{T(t)\}_{t\geq 0}$ is called the semigroup generated by $-A$, and it is said to be compact if $\overline{T(t)B}$ is compact for all $t > 0$ and bounded $B \subset \overline{D(A)}$ (i.e. the $T(t)$ are compact maps for $t > 0$), while $\{T(t)\}_{t\geq 0}$ is called equicontinuous if the family of functions $\{T(\cdot)x: x \in B\}$ is equicontinuous at every $t > 0$, for all bounded $B \subset \overline{D(A)}$. Let us note in passing that $\{T(t)\}_{t\geq 0}$ is compact iff $\{T(t)\}_{t\geq 0}$ is equicontinuous and J_λ is a compact map for some (or, equivalently, for all) $\lambda > 0$; see [10].

Proofs for all facts mentioned above without reference can be found in [3] or [4].

(ii) Let us also provide some facts about multivalued maps; proofs, if not given here, can be found in [15]. We call an F: $J = [0, a] \to 2^X \setminus \emptyset$ measurable if $F^{-1}(V) := \{t \in J: F(t) \cap V \neq \emptyset\}$ is a Lebesgue measurable subset of J, for every open $V \subset X$. If this holds and X is separable then F has a measurable selection, i.e. $f(t) \in F(t)$ a.e. on J for some measurable f: $J \to X$.

Given a nonempty subset Ω of a Banach space and a multivalued map F: $\Omega \to 2^X \setminus \emptyset$, we say that F is usc if $F^{-1}(A)$ is closed in Ω for all closed $A \subset X$, and F is ϵ-δ-usc if for every $x_0 \in \Omega$ and $\epsilon > 0$ there is $\delta = \delta(x_0, \epsilon) > 0$ such that

$$F(x) \subset F(x_0) + B_\epsilon(0) \qquad \text{for all } x \in B_\delta(x_0) \cap \Omega.$$

In general, usc is stronger than ϵ-δ-usc, but both concepts coincide if F has compact values; let us also note that compactness of $\mathrm{gr}(F)$ implies that F is usc with compact values. In applications one also has to consider multivalued maps having only weakly compact values; as a prototype one may think of the following situation: $X = L^p(\Omega)$ with $\Omega = [a, b] \subset \mathbb{R}$, $p \in [1, \infty)$ and F: $X \to 2^X \setminus \emptyset$ given by $F(u) = \{w \in X: w(x) \in \mathrm{Sgn}(u(x)) \text{ a.e. on } \Omega\}$, where $\mathrm{Sgn}(\rho) = \rho/|\rho|$ if $\rho \neq 0$ and $\mathrm{Sgn}(0) = [-1, 1]$. In such cases another concept is more natural. We call F weakly usc if $F^{-1}(A)$ is closed for all weakly closed $A \subset X$. Evidently usc is stronger than weakly usc and simple examples show that a weakly usc F with compact convex values may fail to be usc. Let us record some additional facts about weakly usc maps which will be useful later on.

PROPOSITION 2: *Let X be a Banach space, $\Omega \neq \emptyset$ a subset of another Banach space and F: $\Omega \to 2^X \setminus \emptyset$ have weakly compact values. Then the following holds:*

(a) *If F is ϵ-δ-usc then F is weakly usc.*

(b) *If the values of F are also convex, then F is weakly usc iff $(x_n) \subset \Omega$ with $x_n \to x_0 \in \Omega$ and $y_n \in F(x_n)$ implies $y_{n_k} \rightharpoonup y_0 \in F(x_0)$ for some subsequence (y_{n_k}) of (y_n).*

Proof: To obtain part (a), let F be ϵ-δ-usc and suppose that F is not weakly usc, i.e. there is $(x_n) \subset \Omega$ with $x_n \to x_0 \in \Omega$ and a weakly closed $A \subset X$ such that $F(x_n) \cap A \neq \emptyset$ for all $n \geq 1$ and $F(x_0) \cap A = \emptyset$. Let $\epsilon := \inf\{\rho(y, A): y \in F(x_0)\}$. We are done if $\epsilon > 0$, since then $(F(x_0) + B_\epsilon(0)) \cap A = \emptyset$, hence $F(x_n) \subset F(x_0) + B_\epsilon(0)$ for all large $n \geq 1$ gives the contradiction $F(x_n) \cap A = \emptyset$ for those n.

If $\epsilon = 0$, we find $y_n \in F(x_0)$ and $z_n \in A$ such that $|y_n - z_n| \to 0$. Since $F(x_0)$ is weakly compact we may assume $y_n \rightharpoonup y_0 \in F(x_0)$, hence also $z_n \rightharpoonup y_0 \in A$ which gives $y_0 \in F(x_0) \cap A$, a contradiction.

Concerning (b) notice that sufficiency is obvious. To prove necessity let us first show that $F(C)$ is weakly compact for every compact $C \subset \Omega$. For this purpose let $\bigcup_{\lambda \in \Lambda} V_\lambda$ be any weakly open covering of $F(C)$. For any $x \in C$, $F(x)$ is then covered by finitely many V_λ, the union of which we denote by V_x. Since F is weakly usc and V_x is weakly open the sets $U_x := \{\tilde{x} \in \Omega: F(\tilde{x}) \subset V_x\}$ are open in Ω and cover C. Hence $C \subset \bigcup_{i=1}^m U_{x_i}$ for certain $x_1, \ldots, x_m \in C$. This yields $F(C) \subset \bigcup_{i=1}^m V_{x_i}$ since $y \in F(C)$ means $y \in F(x)$ for some $x \in C$ and $x \in U_{x_i}$ for some i. Therefore, $F(C)$ is weakly compact since $\bigcup_{i=1}^m V_{x_i}$ is the union of finitely many V_λ.

Let $(x_n) \subset \Omega$ with $x_n \to x_0 \in \Omega$ and $y_n \in F(x_n)$. Then $F(\overline{\{x_n: n \geq 1\}})$ is weakly compact, hence $y_{n_k} \rightharpoonup y_0$ for some subsequence. Suppose $y_0 \notin F(x_0)$. By Mazur's Theorem we find $x^* \in X^*$ such that $x^*(y) \leq r$ on $F(x_0)$ and $x^*(y_0) \geq r + 2\delta$ with some $r \in \mathbb{R}$ and $\delta > 0$. Then $F(x_{n_k}) \cap A \neq \emptyset$ for all large $k \geq 1$ with the weakly closed set $A = \{y \in X: x^*(y) \geq r + \delta\}$ implies $F(x_0) \cap A \neq \emptyset$, a contradiction. ∎

In subsequent proofs we shall also use the following fixed point result.

LEMMA 1: *Let X be a Banach space, $\emptyset \neq D \subset X$ compact convex and $F: D \to 2^D \setminus \emptyset$ usc with closed contractible values. Then F has a fixed point.*

This is a special case of the Corollary given in [18], where the values of F are only assumed to be compact R_δ-sets.

(iii) We also need the following criterion for weak relative compactness in $L^1_X(J)$.

LEMMA 2: *Let X be a Banach space, $J = [0, a] \subset \mathbb{R}$ and $W \subset L^1_X(J)$ be uniformly integrable. Suppose that there exist weakly relatively compact sets $C(t) \subset X$ such that $w(t) \in C(t)$ a.e. on J, for all $w \in W$. Then W is weakly relatively compact in $L^1_X(J)$.*

This is Corollary 2.6 in [17] specialized to Lebesgue measure.

3. The compact case

We consider initial value problem (1) in the situation when $-A$ generates a compact semigroup. In this case we can immediately find a compact convex subset of $C_X(J)$ which is invariant under $G = S \circ \mathrm{Sel}$. This is due to the following result.

LEMMA 3: *Let X be a real Banach space, A: $X \to 2^X$ be m-accretive such that $-A$ generates a compact semigroup and let $W \subset L^1_X(J)$ be uniformly integrable. Then $S(W)$ is relatively compact in $C_X(J)$.*

This is Theorem 2.3.3 in [28] which is based on Theorem 2 in [2], where W is of the special type $W = \{w \in L^1_X(J)\colon |w(t)| \le \varphi(t)$ a.e. on $J\}$ with $\varphi \in L^1(J)$.

Now we are able to prove

THEOREM 1: *Let X be a real Banach space and A: $D(A) \subset X \to 2^X \setminus \emptyset$ be m-accretive such that $-A$ generates a compact semigroup. Let $J = [0, a] \subset \mathbb{R}$ and F: $J \times \overline{D(A)} \to 2^X \setminus \emptyset$ with closed convex values be such that $F(\cdot, x)$ has a strongly measurable selection for every $x \in \overline{D(A)}$ and (3) is satisfied. Then (1) has an integral solution for every $x_0 \in \overline{D(A)}$, if also one of the following conditions holds:*

(a) *X^* is uniformly convex, $F(t, \cdot)$ is weakly usc for all $t \in J$.*
(b) *X^* is strictly convex, F has compact values, $F(t, \cdot)$ is usc for all $t \in J$.*

Proof: 1. Let us first show that F admits an extension $\tilde{F}$: $J \times X \to 2^X \setminus \emptyset$ having the same properties as F, such that the solution set of (1) remains the same if F is replaced by $\tilde{F}$. This can be achieved if we let $\tilde{F}(t, x) = F(t, Px)$ on $J \times X$ with P: $X \to \overline{D(A)}$ given by $Px = J_{\lambda(x)}x$, where we let $\lambda(x) = \rho(x, \overline{D(A)})$ on X and $J_0 x := x$ on $\overline{D(A)}$.

Evidently, $\tilde{F}$ has the same properties as F if P is continuous with $|Px| \le c_1 + c_2|x|$ on X, for some c_1, $c_2 > 0$. To prove continuity of P, let $(x_n) \subset X$ with $x_n \to x_0$, $\lambda_n := \lambda(x_n)$ and $\lambda_0 := \lambda(x_0)$. Then $\lambda_n \to \lambda_0$ and if $\lambda_0 = 0$ we have $x_0 \in \overline{D(A)}$, hence

$$|Px_n - Px_0| = |J_{\lambda_n} x_n - x_0| \le |x_n - x_0| + |J_{\lambda_n} x_0 - x_0| \to 0.$$

If $\lambda_0 > 0$ then $\lambda_n > 0$ for all large n, hence

$$\begin{aligned} |Px_n - Px_0| &\le |x_n - x_0| + |J_{\lambda_n} x_0 - J_{\lambda_0} x_0| \\ &\le |x_n - x_0| + \left|1 - \frac{\lambda_n}{\lambda_0}\right| (|x_0| + |J_{\lambda_0} x_0|) \to 0, \end{aligned}$$

where we used the resolvent identity to get the last inequality. To get the estimate for P, fix $\hat{x} \in D(A)$ and $\hat{y} \in A\hat{x}$. Then

$$|Px| \le |x - \hat{x}| + |J_{\lambda(x)}\hat{x}| \le |x - \hat{x}| + \lambda(x)|\hat{y}| + |\hat{x}| \le c_1 + c_2|x| \quad \text{on } X,$$

where $c_1 := |\hat{x}|(2 + |\hat{y}|)$ and $c_2 := 1 + |\hat{y}|$.

Therefore, in the subsequent steps, we may assume that F is defined on $J \times X$; notice that every integral solution u of (1) with $\tilde{F}$ instead of F satisfies $u(J) \subset \overline{D(A)}$, hence u is in fact a solution of the original problem.

2. To get a fixed point of $G = S \circ \mathrm{Sel}$, let us first show $\mathrm{Sel}(u) \ne \emptyset$ for every $u \in C_X(J)$. For this purpose let $u \in C_X(J)$, u_n be step-functions with $|u - u_n|_0 \to 0$ and w_n be strongly measurable selections of $F(\cdot, u_n(\cdot))$. By (3), $\{w_n\colon n \ge 1\} \subset L^1_X(J)$ is uniformly integrable. Moreover, the w_n satisfy $w_n(t) \in C(t) := F(t, \overline{\{u_n(t)\colon n \ge 1\}})$ and the sets $C(t)$ are weakly compact by Proposition 2(b), since F has weakly compact values; notice that X is reflexive in case (a). Therefore we may assume $w_n \rightharpoonup w$ in $L^1_X(J)$ due to Lemma 2. By Mazur's theorem there are $\overline{w}_n \in \mathrm{conv}\{w_k\colon k \ge n\}$ such that $\overline{w}_n \to w$ in $L^1_X(J)$, hence $\overline{w}_{n_k}(t) \to w(t)$ a.e. on J for some subsequence $(\overline{w}_{n_k})$. To conclude $w(t) \in F(t, u(t))$ a.e. on J we argue as follows. Let $t \in J$ be such that $w_n(t) \in F(t, u_n(t))$ for all $n \ge 1$ and $\overline{w}_{n_k}(t) \to w(t)$. Given $x^* \in X^*$ and $\epsilon > 0$, we have $x^*(w_n(t)) \in x^*(F(t, u(t))) + (-\epsilon, \epsilon)$ for all large n, hence the same inclusion holds for $x^*(\overline{w}_{n_k}(t))$ for all large k; notice that $x^* \circ F(t, \cdot)$ is usc with compact convex values. Hence $x^*(w(t)) \in x^*(F(t, u(t))$ for all $x^* \in X^*$ which implies $w(t) \in F(t, u(t))$ a.e. on J, since F has closed convex values. Thus we have $\mathrm{Sel}(u) \ne \emptyset$; in fact the same argument (with $u_n \in C_X(J)$ instead of step-functions) together with Proposition 2(b) also shows that $\mathrm{Sel}\colon C_X(J) \to 2^{L^1_X(J)} \setminus \emptyset$ is weakly usc with weakly compact values.

3. We let $K_0 = \{u \in C_X(J)\colon |u(t)| \le \psi(t) \text{ on } J\}$, where ψ is the solution of

$$\psi' = c(t)(1 + \psi) \text{ a.e. on } J, \quad \psi(0) = \max\{|T(t)x_0|\colon t \in J\}.$$

Then $K_0 \subset C_X(J)$ is closed bounded convex such that $G(K_0) \subset K_0$. Consequently $K\colon = \overline{\mathrm{conv}}G(K_0) \subset C_X(J)$ is compact convex by Lemma 3 and $G(K) \subset K$, hence $G\colon K \to 2^K \setminus \emptyset$ by the first step. To see that the values of G are contractible let $C = G(u)$ for some $u \in K$, fix $\bar{w} \in \mathrm{Sel}(u)$ and define $h\colon [0,1] \times C \to C$ by

$$h(s, v)(t) = \begin{cases} v(t) & \text{if } t \in [0, sa], \\ \bar{u}(t; sa, v(sa)) & \text{if } t \in (sa, a], \end{cases}$$

where $\bar{u}(\cdot; t_0, x_0)$ is the solution of $u' \in -Au + \bar{w}(t)$ on $[t_0, a]$, $u(t_0) = x_0$. Notice that h maps into C, since $v = Sw$ for some $w \in \mathrm{Sel}(u)$, hence $h(s,v) = S\tilde{w}$ with $\tilde{w} := w\chi_{[0,sa]} + \bar{w}\chi_{(sa,a]} \in \mathrm{Sel}(u)$. Evidently $h(0,v) = S\bar{w}$ and $h(1,v) = v$ on C, and h is continuous due to the continuous dependence of $\bar{u}(\cdot; t_0, x_0)$ on $(t_0, x_0) \in [0, a) \times \overline{D(A)}$.

Therefore, Lemma 1 yields a fixed point of G if we are able to prove that $\mathrm{gr}(G)$ is compact. Evidently it suffices to show that $\mathrm{gr}(G)$ is closed. Let $(u_n) \subset K$ with $u_n \to u$ and $v_n \in G(u_n)$ with $v_n \to v$, hence $v_n = Sw_n$ for some $w_n \in \mathrm{Sel}(u_n)$. By the previous step we may assume $w_n \rightharpoonup w \in \mathrm{Sel}(u)$. Now if (a) holds we immediately obtain $v = Sw \in G(u)$ by Proposition 1(a). If (b) is satisfied let $t \in J$ be such that $w_n(t) \in F(t, u_n(t))$ for all $n \geq 1$. Then $w_n(t) \in F(t, u(t)) + B_\epsilon(0)$ for all $n \geq n_\epsilon$, hence $\{w_n(t)\colon n \geq 1\}$ is relatively compact for those $t \in J$. Therefore we get $v = Sw \in G(u)$ by Proposition 1(b). ∎

Let us give some complementary

Remarks: 1. We can also obtain a local version of Theorem 1 as follows: Let F be defined on $J \times D_r$ with $D_r = B_r(x_0) \cap \overline{D(A)}$ and suppose that the corresponding assumptions of Theorem 1 (a) or (b) hold. Then Theorem 1 applies to $\tilde{F}$ given by $\tilde{F}(t,x) = F(t, P(R(x)))$, where P is as in step 1 of the last proof and R is the radial retraction onto $\overline{B}_\delta(x_0)$ with $\delta > 0$ such that $P(\overline{B}_\delta(x_0)) \subset B_r(x_0)$. This yields an integral solution of (1) with $\tilde{F}$ which is a local integral solution of (1) with F, since $F(t,x) = \tilde{F}(t,x)$ on $J \times (B_\delta(x_0) \cap \overline{D(A)})$. Such a local version of part (a) of Theorem 1 comes close to Theorem 3.3.1 in [28]; there it is also assumed (in addition to the conditions imposed in Theorem 1(a)) that X is separable and the $F(\cdot, x)$ are measurable.

If the maps $F(\cdot, x)$ are strongly measurable, i.e. they are given as the a.e.-limit of step-multis with closed values (see §3.3 in [15]), then (3) can be replaced by $\rho(0, F(t,x)) \leq c(t)(1 + |x|)$ on $J \times \overline{D(A)}$ with $c \in L^1(J)$. This follows by application of Theorem 1 to $F_0(t,x) := F(t,x) \cap c(t)(1+|x|)\overline{B}_1(0)$ instead of F. Hence we get at least one global solution, but notice that (1) may have other local solutions as well.

2. In the situation of Theorem 1(b) the extra condition on X^* can be dropped if we consider single-valued perturbations. To see this, notice that the only step where the additional property of X^* came into play was to show that $\mathrm{gr}(G)$ is closed. Now for single-valued $F = \{f\}$ the regularity assumptions become strong measurability w.r. to t and continuity w.r. to x. Therefore, $u_n \to u$ in $C_X(J)$ and $w_n \in \mathrm{Sel}(u_n)$ imply $w_n = f(\cdot, u_n(\cdot)) \to w = f(\cdot, u(\cdot))$ a.e. on J. By the dominated convergence theorem we get $w_n \to w$ in $L^1_X(J)$, hence $Sw_n \to Sw$

and we are done.

In the multivalued case the extra condition on X^* cannot be removed, as shown by the following counter-example.

Example 1: (a) We start with a two-dimensional example in which $w_n \rightharpoonup w$ in $L^1_X(J)$ and $Sw_n \to v$ in $C_X(J)$ do not imply $Sw = v$.

Let $X = \mathbb{R}^2$ with norm $|x|_0 = \max\{|x_1|, |x_2|\}$, $z = (1, -1)$ and $A: X \to 2^X \setminus \emptyset$ be given by

$$Ax = \begin{cases} \{-z\} & \text{if} \quad x_1 < x_2, \\ \{(s, \varphi(s)):\ -1 \le s \le 1\} & \text{if} \quad x_1 = x_2, \\ \{z\} & \text{if} \quad x_1 > x_2, \end{cases}$$

where $\varphi: [-1, 1] \to [-1, 1]$ is continuous, decreasing with $\varphi(-1) = 1$ and $\varphi(1) = -1$. Then it is not difficult to check that A is m-accretive for every choice of such φ; the details can be found in [13] p. 295f, where these particular operators were used to show that the generator of a semigroup may not be uniquely determined.

Let $J = [0, 1]$, $r_n(t) = \mathrm{sgn}(\sin(2^n \pi t))$ be the Rademacher functions (with $\mathrm{sgn}(0) = 1$) and define $(w_n) \subset L^1_X(J)$ by $w_n(t) = r_n(t)z$ on J. Then $w_n(t) \in \{-z, z\}$ on J and $w_n \rightharpoonup 0$ as $n \to \infty$. Due to $\pm z \in A0$, the initial value problems

$$u' \in -Au + w_n(t) \quad \text{on } J, \quad u(0) = 0$$

have the classical solution $u = 0$. Hence $Sw_n = 0$ for all $n \ge 1$, since classical solutions are also integral solutions. So we have $w_n \rightharpoonup 0$ and $Sw_n \to 0$.

Nevertheless, $S0 \neq 0$ unless $\varphi(0) = 0$. Notice that $\varphi(\mu) = \mu$ for some unique $\mu \in (-1, 1)$, hence $(\mu, \mu) \in Ax$ for all $x \in X$ with $x_1 = x_2$. Therefore, the solution of $u' \in -Au$ on J, $u(0) = 0$ is given by $u(t) = (-\mu t, -\mu t)$ on J, hence $u \neq 0$ if $\varphi(0) \neq 0$.

(b) We are now able to define an m-accretive operator A and a compact convex $C \subset X$ such that $S(W)$ is not closed, where

$$W = \{w \in L^1_X(J): w(t) \in C \text{ a.e. on } J\}.$$

For this purpose let $X = \mathbb{R}^4$, equipped with the max-norm, and define $A: X \to 2^X \setminus \emptyset$ as $A = A_1 \times A_2$, where the A_k correspond to functions φ_k and are given as in part (a). As φ_1, φ_2 we choose (see Figure 1)

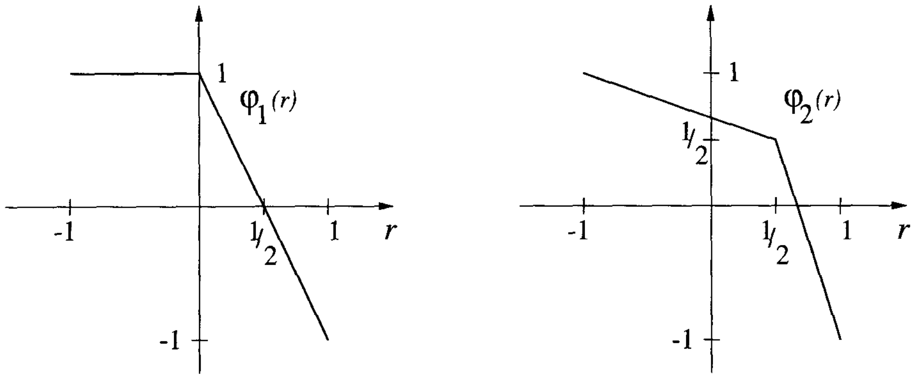

Figure 1.

$$\varphi_1(r) = \begin{cases} 1 & \text{if } \; -1 \le r \le 0 \\ 1-2r & \text{if } \; 0 < r \le 1 \end{cases} \quad \text{and } \varphi_2(r) = \begin{cases} \frac{2-r}{3} & \text{if } \; -1 \le r \le \frac{1}{2} \\ 2-3r & \text{if } \; \frac{1}{2} < r \le 1. \end{cases}$$

Given $e = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, we let $w_n(t) = r_n(t)e$ on $J = [0,1]$ with the r_n from above. For $n \ge 1$, let $u_n = Sw_n$ with initial value $u_n(0) = 0$. Since all u_n are Lipschitz continuous integral solutions and X has the Radon-Nikodym property, it follows that the u_n are strong solutions; see Theorem 7.7 in [4]. By definition of A_k outside the diagonal $\Delta = \{(s,s)\colon s \in \mathbb{R}\}$ and the choice of e it follows that the solution remains in $\Delta \times \Delta$, hence

$$u_n'(t) = \Big(-\mu_1(r_n(t)), -\mu_1(r_n(t)), -\mu_2(r_n(t)), -\mu_2(r_n(t)) \Big) \quad \text{a.e. on } J,$$

where

$$\{(\mu_k(s), \mu_k(s))\} = \Big[\mathrm{gr}(\varphi_k) - s(\tfrac{1}{2}, -\tfrac{1}{2}) \Big] \cap \Delta \quad \text{for } k = 1,2.$$

Elementary calculations show that

$$\mu_1(s) = \frac{1}{3} - \frac{s}{6} \quad \text{and} \quad \mu_2(s) = \frac{1}{2} - \frac{|s|}{4} \quad \text{for } s \in [-1,1].$$

Therefore, $u_n(t) = -ty + (v_n(t), v_n(t), 0, 0)$ on J with

$$y = \Big(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4} \Big) \quad \text{and} \quad v_n(t) = \frac{1}{6} \int_0^t r_n(\tau) d\tau.$$

Consequently, $u_n(t) \to -ty$ uniformly on J. Moreover, $(u_n) \subset S(W)$ if we let $C = \mathrm{conv}\{-e, e\}$ and $W = \{w \in L^1_X(J)\colon w(t) \in C \text{ a.e. on } J\}$. Suppose that $u(t) = -ty$ satisfies $u = Sw$ for some $w \in W$. Then

$$u'(t) \in -Au(t) + w(t) \quad \text{a.e. on } J, \quad u(0) = 0,$$

which means $y + r(t)e \in Au(t)$ a.e. on J with a measurable $r\colon J \to [-1,1]$. By definition of A this yields the contradiction

$$y + \nu e \in \mathrm{gr}(\varphi_1) \times \mathrm{gr}(\varphi_2) \quad \text{for some } \nu \in [-1,1].$$

(c) Based on the previous part we obtain the following example for non-existence. Let A, y, e and C be as in (b) and define $F\colon J \times X \to 2^X \setminus \emptyset$ by

$$F(t,x) = \begin{cases} R(\alpha_k t)e & \text{if } 1/(k+1) < |x+ty| < 1/k, \\ C & \text{otherwise,} \end{cases}$$

where $\alpha_k = k(k+1)$ and $R(s) = \mathrm{Sgn}(\sin(\pi s))$ with $\mathrm{Sgn}(\rho) = \rho/|\rho|$ for $\rho \neq 0$ and $\mathrm{Sgn}(0) = [-1,1]$. Evidently F is usc and bounded with compact convex values.

Assume that (1) with $J = [0,1]$ and $x_0 = 0$ has an integral solution u on J. As in (b) it follows that $u(\cdot) \in \Delta \times \Delta$ and $u(t) = -ty$ is not possible due to $F(t,-ty) \equiv C$.

Let $\psi(t) = |u(t) + ty|$ on J. Due to continuity of ψ, $\psi(0) = 0$ and $\psi \not\equiv 0$ there exist $\tau \in (0,1]$ and a large $k \geq 1$ such that $\psi(\tau) = 1/k$ and $\psi(t) < 1/k$ on $[0,\tau)$. Let $\sigma = \max\{t \in [0,\tau]\colon \psi(t) \leq 1/(k+1)\}$. Then

$$\psi(\sigma) = \frac{1}{k+1}, \quad \psi(\tau) = \frac{1}{k} \quad \text{and} \quad \frac{1}{k+1} < \psi(t) < \frac{1}{k} \quad \text{on } (\sigma,\tau).$$

Consequently $F(t,u(t)) = R(\alpha_k t)e$ on (σ,τ), which implies

$$u' \in -Au + w(t)e \quad \text{a.e. on } [\sigma,\tau] \quad \text{with } w(t) = r_0(\alpha_k t) \text{ on } [\sigma,\tau].$$

As in the previous step this yields

$$u'(t) = \Big(-\frac{1}{3} + \frac{r_0(\alpha_k t)}{6}, -\frac{1}{3} + \frac{r_0(\alpha_k t)}{6}, -\frac{1}{4}, -\frac{1}{4} \Big) \quad \text{a.e. on } [\sigma,\tau],$$

hence

$$u(\tau) + \tau y = u(\sigma) + \sigma y + \Big(\frac{1}{6} \int_\sigma^\tau r_0(\alpha_k t)dt, \frac{1}{6} \int_\sigma^\tau r_0(\alpha_k t)dt, 0, 0 \Big)$$

and therefore the contradiction

$$\frac{1}{k} = \psi(\tau) \leq \psi(\sigma) + \frac{1}{6} \Big| \int_\sigma^\tau r_0(\alpha_k t)dt \Big| \leq \frac{1}{k+1} + \frac{1}{6\alpha_k} < \frac{1}{k}.$$

Thus (1) has no integral solution for this special choice of A, F and x_0.

4. The equicontinuous case

In the important special case when X is a Hilbert space and $A = \partial\varphi$ is the subdifferential of a proper convex lsc function $\varphi\colon D_\varphi \subset X \to \mathbb{R}$, the semigroup generated by $-A$ is always equicontinuous and it is compact iff φ has compact sublevel sets, i.e. $\{x \in X\colon |x|^2 + \varphi(x) \le r\}$ is compact for all $r > 0$; see e.g. p. 42 in [28]. This is one motivation to consider initial value problem (1) in the situation when the semigroup generated by $-A$ is only equicontinuous. Instead of compactness of the resolvents $(I+\lambda A)^{-1}$ we then impose a compactness condition on F, namely

$$\begin{gathered}\beta(F(t,B)) \le k(t)\beta(B)\\ \text{a.e. on } J \text{ for all bounded } B \subset D \text{ with some } k \in L^1(J).\end{gathered} \tag{4}$$

Here $\beta(\cdot)$ denotes the Hausdorff-measure of noncompactness; see §9.2 in [15]. To get existence of integral solutions in this situation we will again use the fixed point approach but this time it is harder to find a compact (convex) invariant set K. In case $A = 0$ this can be achieved using the estimate

$$\beta(\{\int_0^t w_k(s)ds\colon k \ge 1\}) \le \int_0^t \beta(\{w_k(s)\colon k \ge 1\})ds \quad \text{on } J,$$

which holds for separable X and $(w_k) \subset L^1_X(J)$ satisfying $|w_k(t)| \le \varphi(t)$ a.e. on J for all $k \ge 1$ with some $\varphi \in L^1(J)$; see e.g. Proposition 9.3 in [15]. Now the idea is to extend this formula to the case $A \ne 0$. This is contained in

LEMMA 4: *Let X be a real Banach space and $A\colon X \to 2^X$ be m-accretive such that $-A$ generates an equicontinuous semigroup. Then the following holds.*

(a) *If $W \subset L^1_X(J)$ is uniformly integrable then $S(W) \subset C_X(J)$ is equicontinuous.*

(b) *If X^* is uniformly convex and $(w_k) \subset L^1_X(J)$ satisfies $|w_k(t)| \le \varphi(t)$ a.e. on J for all $k \ge 1$ with some $\varphi \in L^1(J)$, then*

$$\beta(\{(Sw_k)(t)\colon k \ge 1\}) \le \int_0^t \beta(\{w_k(s)\colon k \ge 1\})ds \quad \text{on } J. \tag{5}$$

We only have to prove part (b), since the first part is Theorem 2.3 in [19]. Let us also note that (b) is known in the special case $\{w_k(s)\colon k \ge 1\} \subset C$ a.e. on J with some compact $C \subset X$; see Theorem 3.1 in [19]. To prove part (b) we need another auxiliary result. Given $\emptyset \ne \Omega \subset X$ let $\beta_\Omega(B)$ be defined by

$$\beta_\Omega(B) = \inf\{r > 0\colon B \subset \bigcup_{i=1}^m B_r(x_i) \text{ for some } m \ge 1 \text{ and } x_1, \dots, x_m \in \Omega\}$$

for bounded $B \subset \Omega$, i.e. the centers of the covering balls are chosen from Ω instead of X. Then $\beta(B) \leq \beta_\Omega(B) \leq 2\beta(B)$ for all bounded $B \subset \Omega$. Moreover, β_Ω has the following representation:

PROPOSITION 3: *Let X be a Banach space and $\emptyset \neq \Omega_n \subset X$ with $\Omega_n \subset \Omega_{n+1}$ for $n \geq 1$ be such that $\beta(\Omega_n \cap A) = 0$ for bounded $A \subset X$ and all $n \geq 1$. Let $\Omega = \overline{\bigcup_{n\geq 1} \Omega_n}$ and $B = \{x_k: k \geq 1\} \subset \Omega$ be bounded. Then*

$$\beta_\Omega(B) = \lim_{n\to\infty} \overline{\lim_{k\to\infty}}\, \rho(x_k, \Omega_n).$$

This is an extension of Proposition 9.2 in [15], where $\Omega := X$ is assumed to be separable and the Ω_n are subspaces of finite dimension. Nevertheless, except for trivial modifications, the proof given there also works in the situation considered here.

Proof of Lemma 4(b): We may assume that $X_0 = \overline{\operatorname{span}}(\bigcup_{n\geq 1} w_k(J))$ is separable, since all w_k are strongly measurable. By Theorem V.2.3 in [16], which applies since X is in particular reflexive, there is a closed separable subspace Y of X, containing X_0, and a linear continuous projection P from X onto Y with $\|P\| = 1$. For bounded $B \subset Y$ we therefore have $\beta(B) = \beta_Y(B)$. Let $Y_n \subset Y$ be finite-dimensional subspaces such that $Y = \overline{\bigcup_{n\geq 1} Y_n}$, let

$$W_n = \{w \in L^1_{Y_n}(J)\colon |w(s)| \leq 2\varphi(s) \text{ a.e. on } J\}$$

and $\Omega_n = \{(Sw)(t)\colon w \in W_n\}$ for fixed $t \in J$, where it suffices to consider $t > 0$.

Let us show that $\beta(\Omega_n) = 0$ for all $n \geq 1$. For every $\epsilon > 0$ there is a closed $J_\epsilon \subset J$ such that $\varphi_{|J_\epsilon}$ is continuous (hence also bounded) and $\int_{J\setminus J_\epsilon} \varphi(t)dt \leq \epsilon/2$. Then

$$\Omega_n^\epsilon := \{(Sv)(t)\colon v = w\chi_{J_\epsilon} \text{ with } w \in W_n\}$$

is relatively compact by the remark behind Lemma 4. Since $x := (Sw)(t) \in \Omega_n$ implies $x^\epsilon := (S(w\chi_{J_\epsilon}))(t) \in \Omega_n^\epsilon$ and $|x - x_\epsilon| \leq \int_{J\setminus J_\epsilon} |w(t)|dt \leq \epsilon$, we have $\Omega_n \subset \Omega_n^\epsilon + B_\epsilon(0)$. This yields $\beta(\Omega_n) \leq \beta(\Omega_n^\epsilon) + \epsilon = \epsilon$ for all $\epsilon > 0$, i.e. $\beta(\Omega_n) = 0$. Hence Proposition 3 applies and therefore

$$\beta(\{(Sw_k)(t)\colon k \geq 1\}) \leq \beta_\Omega(\{(Sw_k)(t)\colon k \geq 1\}) = \lim_{n\to\infty} \overline{\lim_{k\to\infty}}\, \rho((Sw_k)(t), \Omega_n),$$

where $\Omega = \overline{\bigcup_{n\geq 1} \Omega_n}$. Now

$$\begin{aligned}\rho((Sw_k)(t), \Omega_n) &= \inf\{|(Sw_k)(t) - (Sw)(t)|\colon w \in W_n\}\\ &\leq \inf\{\int_0^t |w_k(s) - w(s)|ds\colon w \in W_n\} = \int_0^t \rho(w_k(s), Y_n)ds;\end{aligned}$$

for the last equality notice that $H(s) := \{x \in Y_n \colon |w_k(s) - x| \le \rho(w_k(s), Y_n)\}$ defines a measurable multivalued map with nonempty values, hence H has a measurable selection w and $|w(s)| \le 2\varphi(s)$ a.e. on J. Finally, using Fatou's Lemma, the dominated convergence theorem and Proposition 3 we get

$$\begin{aligned}\beta(\{(Sw_k)(t)\colon k \ge 1\}) &\le \int_0^t \lim_{n\to\infty} \overline{\lim_{k\to\infty}}\, \rho(w_k(s), Y_n)ds\\ &= \int_0^t \beta_Y(\{w_k(s)\colon k \ge 1\})ds\\ &= \int_0^t \beta(\{w_k(s)\colon k \ge 1\})ds. \qquad \blacksquare\end{aligned}$$

Now we can obtain the main result in this section.

THEOREM 2: *Let X be a real Banach space such that X^* is uniformly convex and $A\colon X \to 2^X$ be m-accretive such that $-A$ generates an equicontinuous semigroup. Let $D := \overline{\mathrm{conv}} D(A)$, $J = [0,a] \subset \mathbb{R}$ and $F\colon J \times D \to 2^X \setminus \emptyset$ with closed convex values satisfying (3) and (4) be such that $F(\cdot, x)$ has a strongly measurable selection for every $x \in D$ and $F(t,\cdot)$ is weakly usc for every $t \in J$. Then (1) has an integral solution for every $x_0 \in \overline{D(A)}$.*

Proof: As in the proof of Theorem 1 we look for a fixed point of $G = S \circ \mathrm{Sel}$, and get a closed bounded convex set $K_0 \subset C_X(J)$ such that $G(K_0) \subset K_0$. We let $K_{n+1} := \overline{\mathrm{conv}} G(K_n)$ for all $n \ge 0$ and $K := \bigcap_{n\ge 0} K_n$. Then we are done if we can show that K is relatively compact; notice that K is then compact convex, and $G\colon K \to 2^K \setminus \emptyset$ is usc with closed contractible values which follows as in the proof of Theorem 1.

Due to Lemma 4(a) we know that K is an equicontinuous subset of $C_X(J)$, hence K is relatively compact if $\beta(K(t)) = 0$ on J, where $K(t) = \{u(t)\colon u \in K\}$. We let $\rho_n(t) = \beta(K_n(t))$ for $n \ge 0$ and $\rho(t) = \beta(K(t))$. Then $\rho_{n+1}(t) \le \beta(\{(Sw)(t)\colon w \in \mathrm{Sel}(K_n)\})$. In order to apply (5) suppose, for the moment, that for given $\epsilon > 0$ there is a sequence $(w_k) \subset \mathrm{Sel}(K_n)$ such that $\beta(\{(Sw)(t)\colon w \in \mathrm{Sel}(K_n)\}) \le 2\beta(\{(Sw_k)(t)\colon k \ge 1\}) + \epsilon$. Then, using also (4), we obtain

$$\rho_{n+1}(t) \le 2\int_0^t \beta(\{w_k(s)\colon k \ge 1\})ds + \epsilon \le 2\int_0^t k(s)\rho_n(s)ds + \epsilon.$$

Since this is true for every $\epsilon > 0$, and $\rho_n(t) \searrow \rho_\infty(t)$ on J, we get $0 \le \rho(t) \le \rho_\infty(t)$ and

$$\rho_\infty(t) \le 2\int_0^t k(s)\rho_\infty(s)ds, \quad \rho_\infty(0) = 0.$$

Evidently, this implies $\rho(t) = 0$ on J.

To finish the proof we have to show that for bounded $B \subset X$ and $\epsilon > 0$ there is a sequence $(x_k) \subset B$ such that $\beta(B) \leq 2\beta(\{x_k\colon k \geq 1\}) + \epsilon$. It suffices to consider $B \subset X$ with $\beta(B) > 0$ and $\epsilon \in (0, \beta(B))$. Let $r = \beta(B) - \epsilon$ and $x_1 \in B$. Then there is $x_2 \in B \setminus B_r(x_1)$ since otherwise $B \subset B_r(x_1)$ gives the contradiction $\beta(B) \leq r$. Given $x_1, \dots, x_m \in B$ with $|x_j - x_k| \geq r$ for $j \neq k$, the same argument yields $x_{m+1} \in B$ such that $|x_j - x_{m+1}| \geq r$ for $j = 1, \dots, m$. By induction we therefore get a sequence $(x_k) \subset B$ with $|x_j - x_k| \geq r$ for all $j \neq k$. Evidently, this implies $\beta(\{x_k\colon k \geq 1\}) \geq r/2$ hence $\beta(B) \leq 2\beta(\{x_k\colon k \geq 1\}) + \epsilon$. ∎

Additional information is contained in the following

Remarks: 3. Theorem 2 is a considerable improvement of Theorem 3.6.1 in [28], where local integral solutions are obtained in the following situation: X separable with X^* uniformly convex, A m-accretive such that $-A$ generates an equicontinuous semigroup, $F\colon [a,b] \times V \to 2^X \setminus \emptyset$ jointly usc with closed convex values satisfying $\beta(F([a,b] \times B)) = 0$ for all bounded $B \subset V$, where V is open in $\overline{D(A)}$ with $x_0 \in V$.

Let us note that Theorem 2 also admits a corresponding local version, say for $F\colon J \times D_r \to 2^X \setminus \emptyset$ with $D_r = B_r(x_0) \cap \overline{\text{conv}} D(A)$; notice that (3) on $J \times D_r$ implies $|u(t) - x_0| < r$ on $[0, b)$ for every integral solution u of (1) if $b > 0$ is sufficiently small.

In Theorem 2 we need the $F(t, \cdot)$ to be defined on a convex set to get the convex compact K; this time we cannot use the map P from the proof of Theorem 1 to extend F to all of $J \times X$, since it is unclear if this extension will satisfy (4). We have not checked whether this can be achieved by other means, since in applications where X^* is uniformly convex, X will usually also have this property and in this case $\overline{D(A)}$ is convex.

4. By the proof of Theorem 2 the set of all integral solutions of (1) is a compact subset of $C_X(J)$, for every $x_0 \in \overline{D(A)}$, in the situation considered there. Consequently, the next example shows that the method which we used does not work without additional assumptions on X^*. More precisely, there exist an m-accretive $A\colon X \to 2^X \setminus \emptyset$ such that $-A$ generates an equicontinuous semigroup and a compact $C \subset X$ such that the set of all integral solutions of

$$u' \in -Au + C \quad \text{on } [0,1], \qquad u(0) = 0 \tag{6}$$

is not relatively compact. The ingredients are taken from a counter-example due to M. Pierre which can be found in [28], p. 224 ff; there it was used to show that a certain sequence of approximate solutions for (6) is not relatively compact.

Notice also that in the subsequent example estimate (5) does not hold.

Example 2: Let $X = \{u \in C_b(\mathbb{R}_+): u(0) = 0\}$ with the sup-norm $|\cdot|_0$. For $u \in X$ we let u^+ be defined by $u^+(x) = \max_{[0,x]} u(s)$; notice that u^+ is increasing with $u^+(x) \geq u(x)$ on $\mathbb{R}_+$. Now we let $D = \{u^+: u \in X\}$ and $A: D \to 2^X \setminus \emptyset$ be given by $Au = \{v - u: v \in X, v^+ = u\}$.

1. Let us show that A is m-accretive such that $-A$ generates an equicontinuous semigroup. For this purpose let us first prove that

$$(7) \qquad (u^+ + \alpha(u - u^+))^+ = u^+ \qquad \text{for all } u \in X \text{ and } \alpha > 0.$$

Let $v = u^+ + \alpha(u - u^+)$. Then $u \leq u^+$ implies $v \leq u^+$ hence also $v^+ \leq u^+$. On the other hand, given $x \geq 0$ there is $\tau \in [0, x]$ with $u^+(x) = u(\tau)$ which yields $u^+(s) = u(\tau)$ for all $s \in [\tau, x]$. This implies

$$v^+(x) \geq v(\tau) = u^+(\tau) + \alpha(u(\tau) - u^+(\tau)) = u^+(x),$$

hence (7) holds.

To show $R(I + \lambda A) = X$ for all $\lambda > 0$, let $w \in X$ be given and

$$v := w^+ + \frac{w - w^+}{\lambda}.$$

Then $u := v^+ \in D$ and $v^+ = w^+ = u$ by (7), hence

$$\frac{w - w^+}{\lambda} \in Au$$

which means $w \in u + \lambda Au$. Moreover, $u = w^+$ is the only solution of $w \in u + \lambda Au$, since $w \in \hat{u} + \lambda A\hat{u}$ with $\hat{u} \in D$ implies $w = \hat{u} + \lambda(\hat{v} - \hat{u})$ for some $\hat{v} \in X$ with $\hat{v}^+ = \hat{u}$, hence $\hat{u} = w^+$ by (7). Therefore, given $\lambda > 0$, $J_\lambda = (I + \lambda A)^{-1}: X \to D$ is well defined and given by $J_\lambda w = w^+$ on X. It remains to show that A is accretive which follows if all J_λ are nonexpansive maps, i.e. $|u^+ - v^+|_0 \leq |u - v|_0$ for all $u, v \in X$. Suppose, on the contrary, that $|u - v|_0 < |u^+(x) - v^+(x)|$ for some $x > 0$, where we may assume $u^+(x) > v^+(x)$. Since $u^+(x) = u(\tau)$ for some $\tau \in [0, x]$ and $v^+(x) \geq v^+(\tau) \geq v(\tau)$, this gives the contradiction $|u(\tau) - v(\tau)| < u^+(x) - v^+(x) \leq u(\tau) - v(\tau)$.

Evidently, $J_\lambda u = u^+$ on X for all $\lambda > 0$ and $u = u^+$ on $\overline{D} = D$ imply

$$T(t)u = \lim_{n \to \infty} J^n_{t/n} u = u \qquad \text{for all } t \geq 0 \text{ and } u \in D.$$

Hence the semigroup generated by $-A$ is given by $T(t) = I_{|D}$ for all $t \geq 0$, which is equicontinuous.

2. Let

$$J = [0,1], \quad J_{n,l} = \left[\frac{l}{n}, \frac{l+1}{n}\right) \quad \text{for } n \geq 1, \quad l = 0, \dots, n-1$$

and $w_n \in L^1_X(J)$ given by $w_n(t) = (-1)^l \varphi$ on $J_{n,l}$, where $\varphi \in X$ is the sawtooth-function defined by $\varphi(x) = \int_0^x \psi(s)ds$ on $\mathbb{R}_+$ with $\psi = \sum_{l \geq 0} (-1)^l \chi_{[2l-1,2l+1)}$. Evidently, $w_n(t) \in C := \{-\varphi, \varphi\}$ on J for all $n \geq 1$. Hence $(w_n) \subset L^1_X(J)$ is weakly relatively compact and we even have $w_n \rightharpoonup 0$. Let $u_n = Sw_n$ be the integral solution of $u' \in -Au + w_n(t)$ on J, $u(0) = 0$. We claim that (u_n) is not relatively compact in $C_X(J)$ and it suffices to show that $(u_{2j}(1)) \subset X$ is not relatively compact.

We will only sketch the proof, since the details require lengthy but elementary calculations. Fix an even $n \geq 2$ and let $t_k = 2k/n$ for $k \geq 0$. Then by induction w.r. to k one can show

$$u(t_k)(x) = \frac{1}{n} \int_0^x (-\psi)^+(s) \chi_{[0,2k+1]}(s) ds \quad \text{on } \mathbb{R}_+, \quad \text{for every } k \geq 0. \tag{8}$$

To get this representation the first step is to solve the initial value problem

$$v' \in -Av + \varphi \quad \text{on } [0, 1/n], \quad v(0) = u(t_k).$$

Since the operator $\tilde{A}$, defined by $\tilde{A}u = Au - \varphi$, is again m-accretive the integral solution of this problem is given by the exponential formula, namely $v(t) = \lim_{m\to\infty} \tilde{J}^m_{t/m} u(t_k)$ where $\tilde{J}_\lambda u = J_\lambda(u + \lambda\varphi) = (u + \lambda\varphi)^+$. Using representation (8) it turns out that $v(t) = (u(t_k) + t\varphi)^+$ for $t \in [0, 1/n]$, hence $u(t_k + 1/n) = v(1/n) = (u(t_k) + \varphi/n)^+$. Then the next step is to take this as the new initial value and to solve $v' \in -Av - \varphi$ on $[1/n, 2/n]$. Evidently $u(t_{k+1}) = v(2/n)$ and a similar argument as above yields

$$u(t_{k+1}) = \left(\left(u(t_k) + \frac{1}{n}\varphi\right)^+ - \frac{1}{n}\varphi\right)^+.$$

Finally, it can be checked by elementary calculations that $u(t_{k+1})$ is again of the type given by (8). Now it is easy to conclude that u_n, for even n, satisfies

$$\left| u_n(1)(x) - \frac{x}{2n} \right| \leq \frac{1}{2n} \quad \text{on } [0, 2n], \quad u_n(1)(x) = 1 \quad \text{on } [2n, \infty).$$

Therefore, $u_{2j}(1)(x) \to 0$ as $j \to \infty$ uniformly on bounded intervals, but $|u_{2j}(1)|_0 = 1$, hence $(u_{2j}(1))$ is not relatively compact which proves the claim.

5. Other special cases

Having clarified that some additional assumption on X^* is needed in general, we are now going to consider two important special cases in which the approach works for general Banach spaces. To achieve this, the crucial step is to check that an estimate like (5) is still valid.

We start with the semilinear case in which we assume that A is also linear. Notice that in this situation $-A$ generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators. Then u is an integral solution of (2) iff

$$u(t) = T(t)x_0 + \int_0^t T(t-s)w(s)ds \quad \text{on } J; \tag{9}$$

see e.g. Theorem 5.7 in [4]. Let $(w_k) \subset L^1_X(J)$ satisfy $|w_k(t)| \leq \varphi(t)$ a.e. on J for all $k \geq 1$ with some $\varphi \in L^1(J)$. Exploitation of (9) together with the estimate mentioned in front of Lemma 4 yields

$$\beta(\{(Sw_k)(t)\colon k \geq 1\}) \leq 2\int_0^t \beta(\{v_k(s)\colon k \geq 1\})ds,$$

where $v_k(s) := T(t-s)w_k(s)$ on $[0,t]$; notice that w.l.o.g. $X_0 = \overline{\operatorname{span}}\bigcup_{k\geq 1} v_k(J)$ is separable, and recall that $\beta_{X_0}(B) \leq 2\beta(B)$ for bounded $B \subset X_0$. In this situation we therefore have

$$\beta(\{(Sw_k)(t)\colon k \geq 1\}) \leq 2\int_0^t \beta(\{w_k(s)\colon k \geq 1\})ds \quad \text{on } J, \tag{10}$$

since accretivity of A implies $|T(t)|_0 \leq 1$ for all $t \geq 0$.

Here we can drop the equicontinuity assumption on the semigroup and have

THEOREM 3: *Let X be a real Banach space and $A\colon D(A) \subset X \to X$ be such that $-A$ generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators. Let $J = [0,a] \subset \mathbb{R}$ and $F\colon J \times X \to 2^X \setminus \emptyset$ with closed convex values satisfying* (3) *and* (4) *be such that $F(\cdot, x)$ has a strongly measurable selection for every $x \in X$ and $F(t, \cdot)$ is weakly usc for every $t \in J$. Then* (1) *has an integral solution for every $x_0 \in X$.*

Proof: Since $-A$ generates a C_0-semigroup, there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $|T(t)|_0 \leq Me^{\omega t}$ on $\mathbb{R}_+$. We may assume $M = 1$ after a change to an equivalent norm, hence $(Ax, x)_+ \geq -\omega|x|^2$ on $D(A)$; see e.g. Chapter 1 in [23]. Let $A_0 := A + \omega I$ and $F_0 := F + \omega I$. Then A_0 and F_0 in place of A and F satisfy the assumptions of Theorem 2. Now notice that the proof of Theorem 2 is also valid in this situation if the set K, as defined there, is relatively compact in $C_X(J)$, and we obtain $\beta(K(t)) \equiv 0$ by the same arguments as given there, since (10) is valid. It remains to prove that K is equicontinuous. Assume, on the contrary, that there are $\epsilon_0 > 0$ and sequences $(u_n) \subset K$, $(t_n), (\bar{t}_n) \subset J$ such that $t_n \to \tau > 0$, $\bar{t}_n \to \tau$ as $n \to \infty$ and $|u_n(t_n) - u_n(\bar{t}_n)| \geq \epsilon_0$ for all $n \geq 1$. We have $u_n = Sw_n$ and $|w_n(t)| \leq \varphi(t)$ a.e. on J for all $n \geq 1$ with some $\varphi \in L^1(J)$, by the definition of K. Exploitation of (9) implies

$$u_n(t) = T(h)u_n(t-h) + \int_{t-h}^{t} T(t-s)w_n(s)ds \text{ for } 0 \leq t-h \leq t \leq a \text{ and } n \geq 1,$$

which yields

$$|u_n(t) - u_n(\bar{t})| \leq |T(t-s)u_n(s) - T(\bar{t}-s)u_n(s)| + \int_s^t \varphi(\rho)d\rho + \int_s^{\bar{t}} \varphi(\rho)d\rho$$

for all $n \geq 1$ and $0 \leq s \leq t, \bar{t} \leq a$. Since $D := \overline{\{u_n(s)\colon n \geq 1\}}$ is compact, the family of maps $\{T(\cdot)x\colon x \in D\}$ is equicontinuous. Therefore, the last inequality gives the contradiction $0 < \epsilon_0 \leq 2\inf\{\int_s^\tau \varphi(\rho)d\rho\colon s \in [0,\tau)\} = 0$. ∎

Now we turn to the case when A is, in addition, single-valued and continuous. We write g instead of $-A$ and allow time-dependence of g, where we assume that g is defined on all of $J \times X$ and is strongly measurable w.r. to t. In this situation we impose the dissipativity condition

$$(11) \qquad \begin{gathered} (g(t,x) - g(t,\bar{x}), x - \bar{x})_- \leq \omega(t)|x-\bar{x}|^2 \\ \text{for all } t \in J, x, \bar{x} \in X \text{ with } \omega \in L^1(J) \end{gathered}$$

as well as the growth condition

$$(12) \qquad |g(t,x)| \leq d(t)(1+|x|) \quad \text{on } J \times X \text{ with } d \in L^1(J).$$

We are then able to obtain strong solutions of the initial value problem

$$(13) \qquad u' \in g(t,u) + F(t,u) \quad \text{a.e. on } J, \quad u(0) = x_0,$$

given that F is as described in Theorem 2. Here u is called strong solution of (13) if u is absolutely continuous and a.e. differentiable such that (13) holds. In analogy to the previous cases, we let Sw denote the (strong) solution of the quasi-autonomous problem

$$(14) \qquad u' = g(t,u) + w(t) \quad \text{a.e. on } J, \quad u(0) = x_0$$

for $w \in L^1_X(J)$.

THEOREM 4: *Let X be a real Banach space, $J = [0,a] \subset \mathbb{R}$ and g: $J \times X \to X$ be strongly measurable w.r. to t, continuous w.r. to x satisfying* (11) *and* (12). *Let F: $J \times X \to 2^X \setminus \emptyset$ with closed convex values satisfying* (3) *and* (4) *be such that $F(\cdot, x)$ has a strongly measurable selection for every $x \in X$ and $F(t,\cdot)$ is weakly usc for every $t \in J$. Then* (13) *has a strong solution for every $x_0 \in X$.*

Proof: 1. Let us first reduce to the case $c(t) \equiv d(t) \equiv k(t) \equiv \omega(t) \equiv 1$. For this purpose define $\varphi \in L^1(J)$ by $\varphi = \max\{1, c, d, k, \omega\}$. The map $t \to \int_0^t \varphi(s)ds$ from J to $\tilde{J} := [0, |\varphi|_1]$ is continuous and strictly increasing. Let ϕ be its inverse and define $\tilde{g}$: $\tilde{J} \times X \to X$ and $\tilde{F}$: $\tilde{J} \times X \to 2^X \setminus \emptyset$ by

$$\tilde{g}(t,x) = \frac{1}{\varphi(\phi(t))} g(\phi(t), x) \quad \text{and} \quad \tilde{F}(t,x) = \frac{1}{\varphi(\phi(t))} F(\phi(t), x) \text{ for } (t,x) \in \tilde{J} \times X.$$

Evidently, u is a strong solution of (13) iff $v(t) := u(\phi(t))$ is a strong solution of (13) with g, F and J replaced by $\tilde{g}$, $\tilde{F}$ and $\tilde{J}$, respectively. Now it is easy to check that $\tilde{g}$ and $\tilde{F}$ satisfy (3), (4), (11) and (12) with $c = d = k = \omega = 1$. Concerning the other properties, notice that $\tilde{g}$ and $\tilde{F}$ are as good as g and F.

In the sequel we denote $\tilde{g}$, $\tilde{F}$ and $\tilde{J}$ by g, F and J, respectively, again.

2. As in step 2 of the proof to Theorem 1 it follows that Sel: $C_X(J) \to 2^{L^1_X(J)} \setminus \emptyset$ is weakly usc with weakly compact convex values. Due to the growth conditions on g and F we find a closed bounded convex $K_0 \subset C_X(J)$ such that $G(K_0) \subset K_0$ for $G = S \circ \text{Sel}$. Moreover, K_0 is equicontinuous since g is bounded on bounded sets.

Now we can repeat the reduction to compact convex $K \subset C_X(J)$ with $G(K) \subset K$, as given in the proof to Theorem 2, if we can show that an estimate like (5) is valid. In fact we have

$$(10) \qquad \beta(\{(Sw_k)(t) \colon k \geq 1\}) \leq 2 \int_0^t \beta(\{w_k(s) \colon k \geq 1\}) ds \quad \text{on } J,$$

for $(w_k) \subset L^1_X(J)$ satisfying $|w_k(t)| \leq \varphi(t)$ a.e. on J for all $k \geq 1$ with some $\varphi \in L^1(J)$. To obtain this inequality it suffices to show relative compactness of $((Sw_k)(t)) \subset X$ for all $t \in (0,a]$, where $(w_k) \subset L^1_X(J)$ satisfies $w_k(t) \in C$ a.e. on J for all $k \geq 1$ with some compact $C \subset X$; remember the proof of Lemma 4(b) and notice that, if we replace Y by X_0 there, we end up with $\int_0^t \beta_{X_0}(\{w_k(s) \colon k \geq 1\})ds$ and then $\beta_{X_0}(B) \leq 2\beta(B)$ for all bounded $B \subset X_0$ yields (10). Given (w_k) of the type mentioned above, we may assume $w_k \rightharpoonup w$ in $L^1_X(J)$ and also $z_k \to z$ in $C_X(J)$, where

$$z_k(t) := \int_0^t w_k(s)ds \quad \text{and} \quad z(t) := \int_0^t w(s)ds \quad \text{on } J.$$

Let $u_k := Sw_k$ and $u := Sw$. Then $\psi(t) := |u_k(t) - z_k(t) - (u(t) - z(t))|$ has

$$\begin{aligned}\psi(t)D^-\psi(t) &= \psi(t)\psi'(t)\\ &= (g(t,u_k(t)) - g(t,u(t)), u_k(t) - (u(t)+e_k(t)))_- \quad \text{a.e. on } J,\end{aligned}$$

where $e_k := z_k - z \to 0$ in $C_X(J)$, hence

$$\psi(t)\psi'(t) \le \psi(t)^2 + (g(t,u(t)+e_k(t)) - g(t,u(t)), u_k(t) - (u(t)+e_k(t)))_+ \text{ a.e. on } J$$

which implies

$$\psi'(t) \le \psi(t) + |g(t,u(t)+e_k(t)) - g(t,u(t))| \quad \text{a.e. on } J.$$

We also have $\psi(0) = 0$, hence Gronwall's Lemma yields

$$\psi(t) \le e^t \int_0^t e^{-s}|g(s,u(s)+e_k(s)) - g(s,u(s))|ds \quad \text{on } J.$$

By continuity of g w.r. to x, (12) and the dominated convergence theorem, the right-hand side tends to zero as $k \to \infty$. Consequently, we have $|u_k - u|_0 \to 0$ as $k \to \infty$ and therefore (10) holds.

In fact we have shown that S: $W \subset L^1_X(X) \to C_X(J)$ is weakly-strongly sequentially continuous, given that W is of the type

$$W = \{w \in L^1_X(J) \colon w(t) \in C \text{ a.e. on } J\}$$

with compact $C \subset X$. Evidently, this implies that $\mathrm{gr}(G_{|K})$ is closed, hence G is usc. By Lemma 1 we get a fixed point u of G, i.e. an integral solution u of (9).

To finish the proof, recall that u is the unique integral solution of (14) for some $w \in \mathrm{Sel}(u)$. On the other hand, initial value problem (14) evidently has a strong solution. Since strong solutions are integral solutions, this implies that u is in fact a strong solution of (13). ∎

Remarks: 5. Theorem 3 can be extended to time-dependent operators $A(t)$ in a straight forward way, if the family $\{A(t)\}_{t\in J}$ generates a strongly continuous evolution system $U(t,s)$, $0 \le s \le t \le a$; see e.g. Chapter 5 in [23] for the definition of an evolution system. In this situation the fixed point approach yields a continuous u: $J \to X$ satisfying

$$u(t) = U(t,0)x_0 + \int_0^t U(t,s)w(s)ds \quad \text{on } J \quad \text{with some } w \in \mathrm{Sel}(u).$$

By the same approach it is also possible to obtain solutions of semilinear functional-differential inclusions. This is done in [22] and the corresponding result is Theorem 2.1 there.

6. If specialized to the single-valued case $F = \{f\}$, Theorem 4 includes the main result in [24], §2. There it is assumed that $f, g: J \times X \to X$ are continuous and bounded such that $\beta(f(J \times B)) \le L\beta(B)$ for all bounded $B \subset X$ and g satisfies (11) with $\omega(t) \equiv \omega$. The latter is an extension of Theorem 2 in [27], which is the local version for compact f; evidently, strong solutions are C^1-solutions in these cases.

The idea to use the special ψ from step 2 of the proof to Theorem 4 to show that $w_n \rightharpoonup w$ implies $Sw_n \to Sw$ is taken from [27].

6. The set of all solutions

As mentioned in Remark 4, the solution set of (1) for fixed $x_0 \in \overline{D(A)}$ is a compact subset of $C_X(J)$ in the situation described in Theorem 2. In fact it is a compact R_δ as we are going to show in this section. Recall that a subset B of a metric space is called compact R_δ, if B is the intersection of a decreasing sequence of compact absolute retracts (AR for short) B_n, i.e. the B_n have the following property: given any metric space Ω, any closed $A \subset \Omega$ and an arbitrary continuous $f: A \to B_n$, there exists a continuous extension $\tilde{f}: \Omega \to B_n$ of f. If f has a continuous extension to some neighborhood V of A only, B_n is called an absolute neighborhood retract (ANR for short). In [20] it was shown that "AR" may be replaced by "contractible" in the definition of compact R_δ; remember that B is contractible if there is $x_0 \in B$ and a continuous $h: [0,1] \times B \to B$ such that

$$h(0,x) = x_0 \quad \text{and} \quad h(1,x) = x \quad \text{on } B.$$

If $B \subset Y$ and the latter holds for some $x_0 \in Y$ and $h: [0,1] \times B \to Y$ then B is said to be contractible in Y, and B is called neighborhood contractible in Y, if B is contractible in every open $V \supset B$. Finally, B is called absolutely neighborhood contractible, if B is neighborhood contractible in Y for every ANR Y which contains B as a closed subset.

To prove that the set of all solutions is a compact R_δ, the following characterization of such sets will be helpful:

LEMMA 5: *Let Ω be a complete metric space, $\beta_0(\cdot)$ denote the Hausdorff-measure of noncompactness in Ω, and let $\emptyset \ne B \subset \Omega$. Then the following statements are equivalent:*

(a) *B is a compact R_δ.*
(b) *$B = \bigcap_{n\geq 1} B_n$ for some decreasing sequence of closed contractible B_n with $\beta_0(B_n) \to 0$.*
(c) *B is compact and absolutely neighborhood contractible.*

Proof: We only have to show "(a) $\Rightarrow$ (b) $\Rightarrow$ (c)", since the last implication "(c) $\Rightarrow$ (a)", which is the most difficult step, forms the main part in [20].

Evidently (a) implies (b), since every AR is contractible. To prove that (c) follows from (b), let Y be any ANR which contains B as a closed subset and let $V \subset Y$ be a neighborhood of B in Y. Since open subsets of ANR's are ANR's too, there is a continuous extension $f: U \to V$ of the identity $I: B \to V$ to some neighborhood U of B in Ω. Then $\beta(B_n) \to 0$ implies $B_n \subset U$ for all large n. Fix such n, let $x_0 \in B_n$ and $h: [0,1] \times B_n \to B_n$ be continuous such that $h(0,x) = x_0$ and $h(1,x) = x$ on B_n. Then $h_0 := f \circ h_{|[0,1]\times B}: [0,1] \times B \to V$ is continuous and satisfies

$$h_0(0,x) = f(x_0) \in V \quad \text{and} \quad h_0(1,x) = x \quad \text{on } B.$$

Hence B is absolutely neighborhood contractible and compactness of B is obvious. ∎

For every $x_0 \in \overline{D(A)}$ we let $M(x_0)$ denote the set of all integral solutions of (1). Then the following holds.

THEOREM 5: *Under the conditions of Theorem 2 the solution set $M(x_0) \subset C_X(J)$ is a compact R_δ. In particular, $M(x_0)$ is connected.*

Proof: 1. Let $x_0 \in \overline{D(A)}$ be given. By the proof of Theorem 2 we already know that $M := M(x_0)$ is compact in $C_X(J)$. Now, to apply Lemma 5, we approximate F by certain $F_n \supset F$. For this purpose let $r_n = 3^{-n}$, $(U_\lambda)_{\lambda\in\Lambda}$ be a locally finite refinement of the open covering $D := \overline{\text{conv}} D(A) \subset \bigcup_{x\in D} B_{r_n}(x)$ and $(\varphi_\lambda)_{\lambda\in\Lambda}$ be a locally Lipschitz continuous partition of unity subordinate to $(U_\lambda)_{\lambda\in\Lambda}$. For every $\lambda \in \Lambda$ let $x_\lambda \in D$ be such that $U_\lambda \subset B_{r_n}(x_\lambda)$. We define F_n by

$$F_n(t,x) = \sum_{\lambda\in\Lambda} \varphi_\lambda(x) C_\lambda(t) \quad \text{on } J \times D \qquad \text{with } C_\lambda(t) := \overline{\text{conv}} F(t, B_{2r_n}(x_\lambda)).$$

Then it is not difficult to show that

$$\begin{aligned} F(t,x) &\subset F_{n+1}(t,x) \subset F_n(t,x) \\ &\subset \overline{\text{conv}} F(t, B_{3r_n}(x) \cap D) \quad \text{on } J \times D \text{ for all } n \geq 1; \end{aligned} \tag{15}$$

see Lemma 2.2 and the proof of Theorem 7.2 in [15], from where this approximation is taken. Let $M_n := M_n(x_0)$ be the solution set of (1) with F_n instead of F. By (15) we know that (M_n) is a decreasing sequence such that $M \subset \bigcap_{n\geq 1} M_n$.

2. We are going to show that $u_n \in M_n$ for all $n \geq 1$ implies $u_{n_k} \to u \in M$ for some subsequence (u_{n_k}) of (u_n). Evidently this yields $M = \bigcap_{n\geq 1} M_n$ but it also implies $\beta_0(M_n) \to 0$, where $\beta_0(\cdot)$ denotes the Hausdorff-measure in $C_X(J)$; notice that we get $\rho_n := \sup_{v\in M_n} \rho(v, M) \to 0$, hence $M_n \subset M + \bar{B}_{\rho_n}(0)$ yields $\beta_0(M_n) \leq \rho_n \to 0$ since M is compact.

Given $u_n = Sw_n \in M_n$ for $n \geq 1$ with $w_n \in F_n(\cdot, u_n(\cdot))$, we first get boundedness of (u_n), since all F_n satisfy

$$\|F_n(t,x)\| \leq c(t)(2 + |x|) \quad \text{on } J \times D \tag{16}$$

by (15) and (3). Hence (w_n) is uniformly integrable and therefore (u_n) is equicontinuous due to Lemma 4(a). Let $\rho(t) = \beta(\{u_n(t): n \geq 1\})$ on J. Application of Lemma 4(b) yields

$$\rho(t) \leq \int_0^t \beta(\{w_n(s): n \geq p\})ds \quad \text{on } J \quad \text{for all } p \geq 1,$$

and exploitation of (15) implies

$$\begin{aligned} \beta(\{w_n(s): n \geq p\}) &\leq \beta(F(s, \{u_n(s): n \geq p\} + B_{3r_p}(0)) \\ &\leq k(s)(\rho(s) + 3r_p) \quad \text{a.e. on } J, \end{aligned} \tag{17}$$

hence Gronwall's Lemma and $p \to \infty$ yields $\rho(t) \equiv 0$. Consequently, $|u_{n_k} - u|_0 \to 0$ for some subsequence (u_{n_k}) and some $u \in C_X(J)$. Since (17) implies $\beta(\{w_n(s): n \geq 1\}) = 0$ a.e. on J we may also assume $w_{n_k} \rightharpoonup w$ by Lemma 2. Now $w \in \mathrm{Sel}(u)$ follows as in step 2 of the proof of Theorem 1, hence $u_{n_k} \to u \in M$.

3. We are done if the $\overline{M}_n$ are contractible, since then Lemma 5 applies with $B = M$ and $B_n = \overline{M}_n$. Fix $n \geq 1$, let g_λ be a strongly measurable selection of $F(\cdot, x_\lambda)$, for every $\lambda \in \Lambda$, and define f by

$$f(t,x) = \sum_{\lambda\in\Lambda} \varphi_\lambda(x) g_\lambda(t) \quad \text{on } J \times D.$$

Then $f(t,x) \in F_n(t,x)$ on $J \times D$ is obvious. Since $(U_\lambda)_{\lambda\in\Lambda}$ is locally finite, $f(\cdot, x)$ is strongly measurable and for every compact $C \subset D$ there exist $\gamma > 0$ and $\delta > 0$ such that

$$|f(t,x) - f(t,\bar{x})| \leq \gamma c(t)|x - \bar{x}| \quad \text{for all } t \in J, \quad x, \bar{x} \in C + B_\delta(0) \tag{18}$$

with $c(\cdot)$ from (3). Therefore, given $(\tau, x) \in J \times \overline{D(A)}$, the local version of Theorem 2 (mentioned in Remark 3) implies the existence of a local integral solution $v = v(\cdot; \tau, x)$ of the initial value problem

$$v' \in -Av + f(t,v) \quad \text{on } [\tau, a], \quad v(\tau) = x. \tag{19}$$

Since (16) also holds with $|f(t,x)|$ instead of $\|F_n(t,x)\|$, every local integral solution of (19) has an extension to all of $[\tau, a]$, and, due to (18), we also have uniqueness. Therefore

$$h(s,u)(t) = \begin{cases} u(t) & \text{if } t \in [0, sa] \\ v(t; sa, u(sa)) & \text{if } t \in (sa, a] \end{cases}$$

defines a function h: $[0,1] \times \overline{M}_n \to \overline{M}_n$ such that $h(0,u) = v(\cdot; 0, x_0)$ and $h(1,u) = u$ on $\overline{M}_n$. It remains to prove that h is continuous in order to show that $\overline{M}_n$ is contractible. For this purpose, let $(s_k, u_k) \in [0,1] \times \overline{M}_n$ with $(s_k, u_k) \to (s, u)$ and $\psi_k(t) := |h(s,u)(t) - h(s_k, u_k)(t)|$ on J. Moreover, let $\gamma, \delta > 0$ be such that (18) holds with $C := \overline{\bigcup_{k \geq 1} u_k(J)}$, and let $K > 0$ be such that $|u|_0 \leq K$ for all $u \in M_1$; such K exists due to (16). Given $\epsilon > 0$, we are going to show

$$\psi_k(t) \leq (2K+5) \exp(\gamma \int_0^t c(\tau) d\tau) \epsilon \quad \text{on } J$$

for all large $k \geq 1$. We consider the case $s_k \leq s$ and $t \in [sa, a]$ only; the remaining cases can be treated by means of similar arguments.

First we find $\eta > 0$ such that $\mu(I) \leq \eta$ implies $\int_I c(\tau) d\tau \leq \epsilon$ for every measurable $I \subset J$. Let $k_\epsilon \geq 1$ be such that $|u - u_k|_0 \leq \epsilon$ and $|s - s_k| \leq \eta / a$ for all $k \geq k_\epsilon$. We have $v(t; s_k a, u_k(s_k a)) = v(t; sa, y_k)$ if we let $y_k := v(sa; s_k a, u_k(s_k a))$, hence exploitation of (18) yields

$$\psi_k(t) = |v(t; sa, u(sa)) - v(t; s_k a, u_k(s_k a))| \leq |u(sa) - y_k| + \gamma \int_{sa}^t c(\tau) \psi_k(\tau) d\tau.$$

Using $u = Sw$ with some $w \in F_n(\cdot, u(\cdot))$ and (16) we get

$$|u(sa) - y_k| \leq |u(s_k a) - u_k(s_k a)| + \int_{s_k a}^{sa} 2(2+K) c(\tau) d\tau,$$

which implies

$$\psi_k(t) \leq (2K+5)\epsilon + \gamma \int_{sa}^t c(\tau) \psi_k(\tau) d\tau \quad \text{for } t \in [sa, a].$$

Consequently, application of Gronwall's Lemma yields the desired estimate, and therefore $|h(s_k, u_k) - h(s,u)|_0 \to 0$ as $k \to \infty$. Hence M is an R_δ-set, and M is connected since all $\overline{M}_n$ are. ∎

Remarks: 7. By inspection of the previous proof it is rather obvious that the solution set of (1) is also a compact R_δ under the assumptions given in Theorem 1, 3 or 4. In the situations as described in Theorem 1 one can make the following simplification in the proof of Theorem 5: Since (16) yields a-priori bounds for all M_n we immediately get compactness of the $\overline{M}_n$ by application of Lemma 3, hence Lemma 5 is not needed then. Part (a) of the result corresponding to Theorem 1 contains Theorem 3.3 in [26].

Concerning properties of the solution set in case $A = 0$ see §7 and §9.3 in [15].

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